

Integrable Noncommutative Sine-Gordon Model

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Abstract

Requiring an infinite number of conserved local charges or the existence of an underlying linear system does not uniquely determine the Moyal deformation of 1+1 dimensional integrable field theories. As an example, the sine-Gordon model may be obtained by dimensional and algebraic reduction from 2+2 dimensional self-dual U(2) Yang-Mills through a 2+1 dimensional integrable U(2) sigma model, with some freedom in the noncommutative extension of this algebraic reduction. Relaxing the latter from $U(2) \rightarrow U(1)$ to $U(2) \rightarrow U(1) \times U(1)$, we arrive at novel noncommutative sine-Gordon equations for a *pair* of scalar fields. The dressing method is employed to construct its multi-soliton solutions. Finally, we evaluate various tree-level amplitudes to demonstrate that our model possesses a factorizable and causal S-matrix in spite of its time-space noncommutativity.

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1 Introduction

In the low-energy limit string theory with D-branes gives rise to noncommutative field theory on the branes when the string propagates in a nontrivial background with an NS-NS two-form B turned on [1]. In particular, if the open string has an $N=2$ worldsheet supersymmetry the tree-level target-space dynamics is described by a noncommutative 2+2 dimensional self-dual Yang-Mills (SDYM) theory [2]. When the target space is filled by n coincident D3-branes the low-energy model is noncommutative $U(n)$ SDYM. This theory is integrable classically [3] and in the sense that it possesses a factorized S-matrix [2].

In the ordinary commutative case it is well known¹ that dimensional reduction of four dimensional SDYM gives rise to many integrable systems in three and two dimensions. This property has been shown to extend also to the noncommutative case² where integrable models in three and in two dimensions have been constructed [6, 7, 8, 9, 10, 11]. In fact, almost all integrable noncommutative models in less than four dimensions³ can be obtained in this fashion.

In 1+1 dimensions, noncommutative models have not yet been considered extensively, primarily because in this situation time is necessarily a noncommutative coordinate, which appears to compromise the causality and unitarity of the theory [18, 19, 20, 21, 17]. However, one may hope that a huge underlying symmetry improves the situation in systems which allow for a Lax-pair formulation and so feature an infinite chain of conserved charges.

In this paper, we are searching a noncommutative generalization of the sine-Gordon system which, as a hallmark of integrability, possesses a well-defined *causal* and *factorized* S-matrix. Furthermore, its equations of motion should admit noncommutative multi-soliton solutions which represent deformations of the well known sine-Gordon solitons.

Previous suggestions for noncommutative sine-Gordon models can be found in [22, 21]. In particular, in [22] a model was proposed which describes the dynamics of a complex scalar field by a couple of equations of motion. These equations were obtained as flatness conditions for a $U(2)$ bidifferential calculus [13] and automatically guarantee the existence of an infinite number of local conserved currents. The same equations were also generated in [17] via a particular dimensional reduction of the noncommutative $U(2)$ SDYM equations in 2+2 dimensions. However, this reduction did *not* work at the level of the action, which turned out to be the sum of two WZW models augmented by a cosine potential. Evaluating tree-level scattering amplitudes it was discovered, furthermore, that this model suffers from acausal behavior and a non-factorized S-matrix, meaning that particle production occurs.

At this point it is important to note that the noncommutative deformation of an integrable equation is a priori not unique, because one may always add terms which vanish in the commutative limit. For the case at hand, for example, different inequivalent ansätze for the $U(2)$ matrices entering the bicomplex construction [13] are possible as long as they all reproduce the ordinary sine-Gordon equation in the commutative limit. It is therefore conceivable that among these possibilities there exists an ansatz (different from the one in [22, 17]) which guarantees the classical integrability of the corresponding noncommutative model. What is already certain is the necessity to introduce *two* real scalar fields instead of one, since in the noncommutative realm the $U(1)$ subgroup of $U(2)$ fails to decouple. What has been missing is a guiding principle towards the “correct” field parametrization.

¹See e.g. [4] and references therein.

²See [5] for reviews on noncommutative field theory.

³See e.g. [12, 13, 14, 15, 16, 17] and references therein.

Since the sine-Gordon model can be obtained by dimensional reduction from 2+2 dimensional SDYM theory via a 2+1 dimensional integrable sigma model [23], and because the latter's noncommutative extension was shown to be integrable in [8], it seems a good idea to construct an integrable generalization of the sine-Gordon equation by starting from the linear system of this integrable sigma model endowed with a time-space noncommutativity. This is the key strategy of this paper. The reduction is performed on the equations of motion first, but it also works at the level of the action, so giving directly the 1+1 dimensional action we are looking for. We interpret this success as an indication that the new field parametrization proposed here is the proper one.

To be more precise, we are going to propose three different parametrizations, by pairs of fields (ϕ_+, ϕ_-) , (ρ, φ) and (h_1, h_2) , all related by nonlocal field redefinitions but all deriving from the compatibility conditions of the underlying linear system [8]. The first two appear in a “Yang gauge” [24] while the third one arises in a “Leznov gauge” [25]. For either field pair in the Yang gauge, the nontrivial compatibility condition reduces to a pair of “noncommutative sine-Gordon equations” which in the commutative limit degenerates to the standard sine-Gordon equation for $\frac{1}{2}(\phi_+ + \phi_-)$ or φ , respectively, while $\frac{1}{2}(\phi_+ - \phi_-)$ or ρ decouple as free bosons. The alternative Leznov formulation has the advantage of producing two polynomial (actually, quadratic) equations of motion for (h_1, h_2) but retains their coupling even in the commutative limit.

With the linear system comes a well-developed technology for generating solitonic solutions to the equations of motion. Here, we shall employ the dressing method [26, 27] to explicitly outline the construction of noncommutative sine-Gordon multi-solitons, directly in 1+1 dimensions as well as by reducing plane-wave solutions of the 2+1 dimensional integrable sigma model [9]. We completely analyze the one-soliton sector where we recover the standard soliton solution as undeformed; noncommutativity becomes palpable only at the multi-soliton level.

It was shown in [2] that the tree-level n -point amplitudes of noncommutative 2+2 dimensional SDYM vanish for $n > 3$, consistent with the vanishing theorems for the $N=2$ string. Therefore, we may expect nice properties of the S-matrix to be inherited by our noncommutative sine-Gordon theory. In fact, a direct evaluation of tree-level amplitudes reveals that, in the Yang as well as the Leznov formulation, the S-matrix is *causal* and no particle production occurs.

An overview of the paper is the following. In section 2 we review the basic construction of the 2+1 dimensional integrable sigma model of [8] through a linear system, for the case of a noncommuting time coordinate. In section 3 we describe its dimensional reduction to the noncommutative integrable sine-Gordon model, both in the Yang and the Leznov formulation. Section 4 is devoted to the construction of solitonic solutions for our model, by way of the iterative dressing approach. The computation of scattering amplitudes is described in section 5. Finally, section 6 contains our conclusions and possible future directions.

2 Noncommutative integrable sigma model in 2+1 dimensions

As has been known for some time, nonlinear sigma models in 2+1 dimensions may be Lorentz-invariant or integrable but not both [23]. Since the integrable variant serves as our starting point for the derivation of the sine-Gordon model and its soliton solutions, we shall present its noncommutative extension [8] in some detail in the present section.

Noncommutative $\mathbb{R}^{2,1}$. Classical field theory on noncommutative spaces may be realized by deforming the ordinary product of classical fields (or their components) to the noncommutative star product

$$(f \star g)(x) = f(x) \exp \left\{ \frac{i}{2} \overleftarrow{\partial}_a \theta^{ab} \overrightarrow{\partial}_b \right\} g(x) \quad , \quad (2.1)$$

with a constant antisymmetric tensor θ^{ab} , where $a, b, \dots = 0, 1, 2$. Specializing to $\mathbb{R}^{2,1}$, we shall use (real) coordinates $(x^a) = (t, x, y)$ in which the Minkowskian metric reads $(\eta_{ab}) = \text{diag}(-1, +1, +1)$. For later use we introduce the light-cone coordinates

$$u := \frac{1}{2}(t + y) \quad , \quad v := \frac{1}{2}(t - y) \quad , \quad \partial_u = \partial_t + \partial_y \quad , \quad \partial_v = \partial_t - \partial_y \quad . \quad (2.2)$$

In view of the future reduction to 1+1 dimensions, we choose the coordinate x to remain commutative, so that the only non-vanishing component of the noncommutativity tensor is

$$\theta^{ty} = -\theta^{yt} =: \theta > 0 \quad . \quad (2.3)$$

Linear system. Consider on noncommutative $\mathbb{R}^{2,1}$ the following pair of linear differential equations [8],

$$(\zeta \partial_x - \partial_u) \Psi = A \star \Psi \quad \text{and} \quad (\zeta \partial_v - \partial_x) \Psi = B \star \Psi \quad , \quad (2.4)$$

where a spectral parameter $\zeta \in \mathbb{C}P^1 \cong S^2$ has been introduced. The auxiliary field Ψ takes values in $U(n)$ and depends on (t, x, y, ζ) or, equivalently, on (x, u, v, ζ) . The $u(n)$ matrices A and B , in contrast, do not depend on ζ but only on (x, u, v) . Given a solution Ψ , they can be reconstructed via⁴

$$A = \Psi \star (\partial_u - \zeta \partial_x) \Psi^{-1} \quad \text{and} \quad B = \Psi \star (\partial_x - \zeta \partial_v) \Psi^{-1} \quad . \quad (2.5)$$

It should be noted that the equations (2.4) are not of first order but actually of infinite order in derivatives, due to the star products involved. In addition, the matrix Ψ is subject to the following reality condition [23]:

$$\mathbf{1} = \Psi(t, x, y, \zeta) \star [\Psi(t, x, y, \bar{\zeta})]^\dagger \quad , \quad (2.6)$$

where ‘ \dagger ’ is hermitian conjugation. The compatibility conditions for the linear system (2.4) read

$$\partial_x B - \partial_v A = 0 \quad , \quad (2.7)$$

$$\partial_x A - \partial_u B - A \star B + B \star A = 0 \quad . \quad (2.8)$$

By detailing the behavior of Ψ at small ζ and at large ζ we shall now “solve” these equations in two different ways, each one leading to a single equation of motion for a particular field theory.

Yang-type solution. We require that Ψ is regular at $\zeta=0$ [28],

$$\Psi(t, x, y, \zeta \rightarrow 0) = \Phi^{-1}(t, x, y) + O(\zeta) \quad , \quad (2.9)$$

which defines a $U(n)$ -valued field $\Phi(t, x, y)$, i.e. $\Phi^\dagger = \Phi^{-1}$. Therewith, A and B are quickly reconstructed via

$$A = \Psi \star \partial_u \Psi^{-1} \Big|_{\zeta=0} = \Phi^{-1} \star \partial_u \Phi \quad \text{and} \quad B = \Psi \star \partial_x \Psi^{-1} \Big|_{\zeta=0} = \Phi^{-1} \star \partial_x \Phi \quad . \quad (2.10)$$

It is easy to see that compatibility equation (2.8) is then automatic while the remaining equation (2.7) turns into [8]

$$\partial_x (\Phi^{-1} \star \partial_x \Phi) - \partial_v (\Phi^{-1} \star \partial_u \Phi) = 0 \quad . \quad (2.11)$$

This Yang-type equation [24] can be rewritten as

$$(\eta^{ab} + v_c \epsilon^{cab}) \partial_a (\Phi^{-1} \star \partial_b \Phi) = 0 \quad , \quad (2.12)$$

⁴Inverses are understood with respect to the star product, i.e. $\Psi^{-1} \star \Psi = \mathbf{1}$.

where ϵ^{abc} is the alternating tensor with $\epsilon^{012}=1$ and $(v_c) = (0, 1, 0)$ is a fixed spacelike vector. Clearly, this equation is not Lorentz-invariant but (deriving from a Lax pair) it is integrable.

One can recognize (2.12) as the field equation for (a noncommutative generalization of) a WZW-like modified $U(n)$ sigma model [23, 29] with the action⁵

$$S_Y = -\frac{1}{2} \int dt dx dy \eta^{ab} \text{tr} \left(\partial_a \Phi^{-1} \star \partial_b \Phi \right) - \frac{1}{3} \int dt dx dy \int_0^1 d\lambda \tilde{v}_\rho \epsilon^{\rho\mu\nu\sigma} \text{tr} \left(\tilde{\Phi}^{-1} \star \partial_\mu \tilde{\Phi} \star \tilde{\Phi}^{-1} \star \partial_\nu \tilde{\Phi} \star \tilde{\Phi}^{-1} \star \partial_\sigma \tilde{\Phi} \right) , \quad (2.13)$$

where Greek indices include the extra coordinate λ , and $\epsilon^{\rho\mu\nu\sigma}$ denotes the totally antisymmetric tensor in \mathbb{R}^4 . The field $\tilde{\Phi}(t, x, y, \lambda)$ is an extension of $\Phi(t, x, y)$, interpolating between

$$\tilde{\Phi}(t, x, y, 0) = \text{const} \quad \text{and} \quad \tilde{\Phi}(t, x, y, 1) = \Phi(t, x, y) , \quad (2.14)$$

and ‘tr’ implies the trace over the $U(n)$ group space. Finally, $(\tilde{v}_\rho) = (v_c, 0)$ is a constant vector in (extended) space-time.

Leznov-type solution. Finally, we also impose the asymptotic condition that $\lim_{\zeta \rightarrow \infty} \Psi = \Psi^0$ with some constant unitary (normalization) matrix Ψ^0 . The large ζ behavior [28]

$$\Psi(t, x, y, \zeta \rightarrow \infty) = (\mathbf{1} + \zeta^{-1} \Upsilon(t, x, y) + O(\zeta^{-2})) \Psi^0 \quad (2.15)$$

then defines a $u(n)$ -valued field $\Upsilon(t, x, y)$. Again this allows one to reconstruct A and B through

$$A = - \lim_{\zeta \rightarrow \infty} (\zeta \Psi \star \partial_x \Psi^{-1}) = \partial_x \Upsilon \quad \text{and} \quad B = - \lim_{\zeta \rightarrow \infty} (\zeta \Psi \star \partial_v \Psi^{-1}) = \partial_v \Upsilon . \quad (2.16)$$

In this parametrization, compatibility equation (2.7) becomes an identity but the second equation (2.8) turns into [8]

$$\partial_x^2 \Upsilon - \partial_u \partial_v \Upsilon - \partial_x \Upsilon \star \partial_v \Upsilon + \partial_v \Upsilon \star \partial_x \Upsilon = 0 . \quad (2.17)$$

This Leznov-type equation [25] can also be obtained by extremizing the action

$$S_L = \int dt dx dy \text{tr} \left\{ \frac{1}{2} \eta^{ab} \partial_a \Upsilon \star \partial_b \Upsilon + \frac{1}{3} \Upsilon \star (\partial_x \Upsilon \star \partial_v \Upsilon - \partial_v \Upsilon \star \partial_x \Upsilon) \right\} , \quad (2.18)$$

which is merely cubic.

Obviously, the Leznov field Υ is related to the Yang field Φ through the non-local field redefinition

$$\partial_x \Upsilon = \Phi^{-1} \star \partial_u \Phi \quad \text{and} \quad \partial_v \Upsilon = \Phi^{-1} \star \partial_x \Phi . \quad (2.19)$$

For each of the two fields Φ and Υ , one equation from the pair (2.7, 2.8) represents the equation of motion, while the other one is a direct consequence of the parametrization (2.10) or (2.16).

⁵which is obtainable by dimensional reduction from the Nair-Schiff action [30, 31] for SDYM in 2+2 dimensions

3 Reduction to noncommutative sine-Gordon

Algebraic reduction ansatz. It is well known that the (commutative) sine-Gordon equation can be obtained from the self-duality equations for SU(2) Yang-Mills upon appropriate reduction from 2+2 to 1+1 dimensions. In this process the integrable sigma model of the previous section appears as an intermediate step in 2+1 dimensions, and so we may take its noncommutative extension as our departure point, after enlarging the group to U(2). In order to avoid cluttering the formulae we suppress the ‘ \star ’ notation for noncommutative multiplication from now on: all products are assumed to be star products, and all functions are built on them, i.e. $e^{f(x)}$ stands for $e_\star^{f(x)}$ and so on.

The dimensional reduction proceeds in two steps, firstly, a factorization of the coordinate dependence and, secondly, an algebraic restriction of the form of the U(2) matrices involved. In the language of the linear system (2.4) the adequate ansatz for the auxiliary field Ψ reads

$$\Psi(t, x, y, \zeta) = V(x) \psi(u, v, \zeta) V^\dagger(x) \quad \text{with} \quad V(x) = \mathcal{E} e^{i\alpha x \sigma_1}, \quad (3.1)$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, \mathcal{E} denotes some constant unitary matrix (to be specified later) and α is a constant parameter. Under this factorization, the linear system (2.4) simplifies to⁶

$$(\partial_u - i\alpha \zeta \text{ad}\sigma_1) \psi = -a \psi \quad \text{and} \quad (\zeta \partial_v - i\alpha \text{ad}\sigma_1) \psi = b \psi \quad (3.2)$$

with $a = V^\dagger A V$ and $b = V^\dagger B V$. Taking into account the asymptotic behavior (2.9, 2.15), the ansatz (3.1) translates to the decompositions

$$\Phi(t, x, y) = V(x) g(u, v) V^\dagger(x) \quad \text{with} \quad g(u, v) \in \text{U}(2), \quad (3.3)$$

$$\Upsilon(t, x, y) = V(x) \chi(u, v) V^\dagger(x) \quad \text{with} \quad \chi(u, v) \in u(2). \quad (3.4)$$

To aim for the sine-Gordon equation, one imposes certain algebraic constraints on a and b (and therefore on ψ). Their precise form, however, is not needed, as we are ultimately interested only in g or χ . Therefore, we instead directly restrict $g(u, v)$ to the form

$$g = \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} = g_+ P_+ + g_- P_- \quad \text{with} \quad g_+ \in \text{U}(1)_+ \quad \text{and} \quad g_- \in \text{U}(1)_- \quad (3.5)$$

and with projectors $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This imbeds g into a $\text{U}(1) \times \text{U}(1)$ subgroup of $\text{U}(2)$. Note that g_+ and g_- do not commute, due to the implicit star product. Invoking the field redefinition (2.19) we infer that the corresponding reduction for $\chi(u, v)$ should be⁷

$$\chi = i \begin{pmatrix} 0 & h^\dagger \\ h & 0 \end{pmatrix} \quad \text{with} \quad h \in \mathbb{C}, \quad (3.6)$$

with the “bridge relations”

$$\begin{aligned} \alpha(h - h^\dagger) &= -g_+^\dagger \partial_u g_+ = g_-^\dagger \partial_u g_- , \\ \frac{1}{\alpha} \partial_v h &= g_-^\dagger g_+ - \mathbf{1} \quad \text{and h.c.} \end{aligned} \quad (3.7)$$

In this way, the $u(2)$ -matrix χ is restricted to be off-diagonal.

We now investigate in turn the consequences of the ansätze (3.3, 3.5) and (3.4, 3.6) for the equations of motion (2.11) and (2.17), respectively.

⁶The adjoint action means $\text{ad}\sigma_1(\psi) = [\sigma_1, \psi]$.

⁷Complex conjugates of scalar functions are denoted with a dagger to remind the reader of their noncommutativity.

Reduction of Yang-type equation. Let us insert the ansatz (3.3) into the Yang-type equation of motion (2.11). After stripping off the V factors one obtains

$$\partial_v(g^\dagger \partial_u g) + \alpha^2(\sigma_1 g^\dagger \sigma_1 g - g^\dagger \sigma_1 g \sigma_1) = 0 \quad . \quad (3.8)$$

Specializing with (3.5) and employing the identities $\sigma_1 P_\pm \sigma_1 = P_\mp$ we arrive at $Y_+ P_+ + Y_- P_- = 0$, with

$$\begin{aligned} Y_+ &\equiv \partial_v(g_+^\dagger \partial_u g_+) + \alpha^2(g_-^\dagger g_+ - g_+^\dagger g_-) = 0 \quad , \\ Y_- &\equiv \partial_v(g_-^\dagger \partial_u g_-) + \alpha^2(g_+^\dagger g_- - g_-^\dagger g_+) = 0 \quad . \end{aligned} \quad (3.9)$$

Since the brackets multiplying α^2 are equal and opposite, it is worthwhile to present the sum and the difference of the two equations:

$$\begin{aligned} \partial_v(g_+^\dagger \partial_u g_+ + g_-^\dagger \partial_u g_-) &= 0 \quad , \\ \partial_v(g_+^\dagger \partial_u g_+ - g_-^\dagger \partial_u g_-) &= 2\alpha^2(g_+^\dagger g_- - g_-^\dagger g_+) \quad . \end{aligned} \quad (3.10)$$

It is natural to introduce the angle fields $\phi_\pm(u, v)$ via

$$g = e^{\frac{i}{2}\phi_+ P_+} e^{-\frac{i}{2}\phi_- P_-} \quad \Leftrightarrow \quad g_+ = e^{\frac{i}{2}\phi_+} \quad \text{and} \quad g_- = e^{-\frac{i}{2}\phi_-} \quad . \quad (3.11)$$

In terms of these, the equations (3.10) read

$$\begin{aligned} \partial_v(e^{-\frac{i}{2}\phi_+} \partial_u e^{\frac{i}{2}\phi_+} + e^{\frac{i}{2}\phi_-} \partial_u e^{-\frac{i}{2}\phi_-}) &= 0 \quad , \\ \partial_v(e^{-\frac{i}{2}\phi_+} \partial_u e^{\frac{i}{2}\phi_+} - e^{\frac{i}{2}\phi_-} \partial_u e^{-\frac{i}{2}\phi_-}) &= 2\alpha^2(e^{-\frac{i}{2}\phi_+} e^{-\frac{i}{2}\phi_-} - e^{\frac{i}{2}\phi_-} e^{\frac{i}{2}\phi_+}) \quad . \end{aligned} \quad (3.12)$$

We propose to call these two equations “the noncommutative sine-Gordon equations”. Besides their integrability (see later sections for consequences) their form is quite convenient for studying the commutative limit. When $\theta \rightarrow 0$, (3.12) simplifies to

$$\partial_u \partial_v(\phi_+ - \phi_-) = 0 \quad \text{and} \quad \partial_u \partial_v(\phi_+ + \phi_-) = -8\alpha^2 \sin \frac{1}{2}(\phi_+ + \phi_-) \quad . \quad (3.13)$$

Because the equations have decoupled we may choose

$$\phi_+ = \phi_- =: \phi \quad \Leftrightarrow \quad g_+ = g_-^\dagger \quad \Leftrightarrow \quad g \in \text{U}(1)_A \quad (3.14)$$

and reproduce the familiar sine-Gordon equation

$$(\partial_t^2 - \partial_y^2) \phi = -4\alpha^2 \sin \phi \quad . \quad (3.15)$$

One learns that in the commutative case the reduction is $\text{SU}(2) \rightarrow \text{U}(1)_A$ since the $\text{U}(1)_V$ degree of freedom $\phi_+ - \phi_-$ is not needed. The deformed situation, however, requires extending $\text{SU}(2)$ to $\text{U}(2)$, and so it is imperative here to keep both $\text{U}(1)$ s and work with *two* scalar fields.

Inspired by the commutative decoupling, one may choose another distinguished parametrization of g , namely

$$g_+ = e^{\frac{i}{2}\rho} e^{\frac{i}{2}\varphi} \quad \text{and} \quad g_- = e^{\frac{i}{2}\rho} e^{-\frac{i}{2}\varphi} \quad , \quad (3.16)$$

which defines angles $\rho(u, v)$ and $\varphi(u, v)$ for the linear combinations $\text{U}(1)_V$ and $\text{U}(1)_A$, respectively. Inserting this into (3.9) one finds

$$\begin{aligned} \partial_v(e^{-\frac{i}{2}\varphi} \partial_u e^{\frac{i}{2}\varphi}) + 2i\alpha^2 \sin \varphi &= -\partial_v[e^{-\frac{i}{2}\varphi} e^{-\frac{i}{2}\rho} (\partial_u e^{\frac{i}{2}\rho}) e^{\frac{i}{2}\varphi}] \quad , \\ \partial_v(e^{\frac{i}{2}\varphi} \partial_u e^{-\frac{i}{2}\varphi}) - 2i\alpha^2 \sin \varphi &= -\partial_v[e^{\frac{i}{2}\varphi} e^{-\frac{i}{2}\rho} (\partial_u e^{\frac{i}{2}\rho}) e^{-\frac{i}{2}\varphi}] \quad . \end{aligned} \quad (3.17)$$

In the commutative limit, this system is easily decoupled to

$$\partial_u \partial_v \rho = 0 \quad \text{and} \quad \partial_u \partial_v \varphi + 4\alpha^2 \sin \varphi = 0 \quad , \quad (3.18)$$

revealing that $\rho \rightarrow \frac{1}{2}(\phi_+ - \phi_-)$ and $\varphi \rightarrow \frac{1}{2}(\phi_+ + \phi_-) = \phi$ in this limit.

It is not difficult to write down an action for (3.9) (and hence for (3.12) or (3.17)). The relevant action may be computed by reducing (2.13) with the help of (3.3) and (3.5). The result takes the form

$$S[g_+, g_-] = S_W[g_+] + S_W[g_-] + \alpha^2 \int dt dy (g_+^\dagger g_- + g_-^\dagger g_+ - 2) \quad , \quad (3.19)$$

where S_W is the abelian WZW action

$$S_W[f] \equiv -\frac{1}{2} \int dt dy \partial_v f^{-1} \partial_u f - \frac{1}{3} \int dt dy \int_0^1 d\lambda \epsilon^{\mu\nu\sigma} \hat{f}^{-1} \partial_\mu \hat{f} \hat{f}^{-1} \partial_\nu \hat{f} \hat{f}^{-1} \partial_\sigma \hat{f} \quad . \quad (3.20)$$

Here $\hat{f}(\lambda)$ is a homotopy path satisfying the conditions $\hat{f}(0) = 1$ and $\hat{f}(1) = f$. Parametrizing g_\pm as in (3.16) and using the Polyakov-Wiegmann identity, the action for ρ and φ reads

$$\begin{aligned} S[\rho, \varphi] = & 2S_{PC}[e^{\frac{i}{2}\varphi}] + 2\alpha^2 \int dt dy (\cos \varphi - 1) + 2S_W[e^{\frac{i}{2}\rho}] \\ & - \int dt dy e^{-\frac{i}{2}\rho} \partial_v e^{\frac{i}{2}\rho} (e^{-\frac{i}{2}\varphi} \partial_u e^{\frac{i}{2}\varphi} + e^{\frac{i}{2}\varphi} \partial_u e^{-\frac{i}{2}\varphi}) \quad , \end{aligned} \quad (3.21)$$

where

$$S_{PC}[f] \equiv -\frac{1}{2} \int dt dy \partial_v f^{-1} \partial_u f \quad . \quad (3.22)$$

In this parametrization the WZ term has apparently been shifted entirely to the ρ field while the cosine-type self-interaction remains for the φ field only. This fact has important consequences for the scattering amplitudes.

It is well known [32, 33, 34] that in ordinary commutative geometry the bosonization of N free massless fermions in the fundamental representation of $SU(N)$ gives rise to a WZW model for a scalar field in $SU(N)$ plus a free scalar field associated with the $U(1)$ invariance of the fermionic system. In the noncommutative case the bosonization of a single massless Dirac fermion produces a noncommutative $U(1)$ WZW model [35], which becomes free only in the commutative limit. Moreover, the $U(1)$ subgroup of $U(N)$ does no longer decouple [36], so that N noncommuting free massless fermions are related to a noncommutative WZW model for a scalar in $U(N)$. On the other hand, giving a mass to the single Dirac fermion leads to a noncommutative cosine potential on the bosonized side [37].

In contrast, the noncommutative sine-Gordon model we propose in this paper is of a more general form. The action (3.19) describes the propagation of a scalar field g taking its value in $U(1) \times U(1) \subset U(2)$. Therefore, we expect it to be a bosonized version of two fermions in some representation of $U(1) \times U(1)$. The absence of a WZ term for φ and the lack of a cosine-type self-interaction for ρ as well as the non-standard interaction term make the precise identification non-trivial however.

Reduction of Leznov-type equation. Alternatively, if we insert the ansatz (3.4) into the Leznov-type equation of motion (2.17) we get

$$\partial_u \partial_v \chi + 2\alpha^2 (\chi - \sigma_1 \chi \sigma_1) + i\alpha [[\sigma_1, \chi], \partial_v \chi] = 0 \quad . \quad (3.23)$$

Specializing with (3.6) this takes the form $Z\sigma_- + Z^\dagger\sigma_+ = 0$ with $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, where

$$Z \equiv \partial_u \partial_v h + 2\alpha^2 (h - h^\dagger) + \alpha \{ \partial_v h, h - h^\dagger \} = 0 \quad . \quad (3.24)$$

The decomposition

$$\chi = i(h_1\sigma_1 + h_2\sigma_2) \quad \Leftrightarrow \quad h = h_1 + i h_2 \quad (3.25)$$

then yields

$$\begin{aligned} \partial_u \partial_v h_1 - 2\alpha \{ \partial_v h_2, h_2 \} &= 0 \quad , \\ \partial_u \partial_v h_2 + 4\alpha^2 h_2 + 2\alpha \{ \partial_v h_1, h_2 \} &= 0 \quad . \end{aligned} \quad (3.26)$$

These two equations constitute an alternative description of the noncommutative sine-Gordon model; they are classically equivalent to the pair of (3.10) or, to be more specific, to the pair of (3.17). For the real fields the “bridge relations” (3.7) read

$$\begin{aligned} 2i\alpha h_2 &= -e^{-\frac{i}{2}\varphi} e^{-\frac{i}{2}\rho} \partial_u (e^{\frac{i}{2}\rho} e^{\frac{i}{2}\varphi}) = e^{\frac{i}{2}\varphi} e^{-\frac{i}{2}\rho} \partial_u (e^{\frac{i}{2}\rho} e^{-\frac{i}{2}\varphi}) \quad , \\ \frac{1}{\alpha} \partial_v h_1 &= \cos \varphi - 1 \quad \text{and} \quad \frac{1}{\alpha} \partial_v h_2 = \sin \varphi \quad . \end{aligned} \quad (3.27)$$

One may “solve” one equation of (3.17) by an appropriate field redefinition from (3.27), which implies already one member of (3.26). The second equation from (3.17) then yields the remaining “bridge relations” in (3.27) as well as the other member of (3.26). This procedure works as well in the opposite direction, from (3.26) to (3.17). The nonlocal duality between (φ, ρ) and (h_1, h_2) is simply a consequence of the equivalence between (2.11) and (2.17) which in turn follows from our linear system (2.4).

The “ h description” has the advantage of being polynomial. It is instructive to expose the action for the system (3.26). Either by inspection or by reducing the Leznov action (2.18) one obtains

$$S[h_1, h_2] = \int dt dy \left\{ \partial_u h_1 \partial_v h_1 + \partial_u h_2 \partial_v h_2 - 4\alpha^2 h_2^2 - 4\alpha h_2^2 \partial_v h_1 \right\} \quad . \quad (3.28)$$

Relation with other noncommutative generalizations of sine-Gordon. The noncommutative generalizations of the sine-Gordon model presented above are expected to possess an infinite number of conservation laws, as they originate from the reduction of an integrable model [7]. It is worthwhile to point out their relation to previously proposed noncommutative sine-Gordon models which also feature an infinite number of local conserved currents.

In [22] an alternative noncommutative version of the sine-Gordon model was proposed. Using the bicomplex approach the equations of motion were obtained as flatness conditions of a bidifferential calculus,⁸

$$\bar{\partial}(G^{-1} \star \partial G) = [R, G^{-1} \star S G]_\star \quad , \quad (3.29)$$

where

$$R = S = 2\alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.30)$$

and G is a suitable matrix in $U(2)$ or, more generally, in complexified $U(2)$. In [22] the G matrix was chosen as

$$G = e^{\frac{i}{2}\sigma_2\phi} = \begin{pmatrix} \cos_\star \frac{\phi}{2} & \sin_\star \frac{\phi}{2} \\ -\sin_\star \frac{\phi}{2} & \cos_\star \frac{\phi}{2} \end{pmatrix} \quad (3.31)$$

⁸This subsection switches to Euclidean space \mathbb{R}^2 , where ∂ and $\bar{\partial}$ are derivatives with respect to complex coordinates.

with ϕ being a complex scalar field. This choice produces the noncommutative equations (all the products are \star -products)

$$\begin{aligned}\bar{\partial}\left(e^{\frac{i}{2}\phi}\partial e^{-\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi}\partial e^{\frac{i}{2}\phi}\right) &= 0 \quad , \\ \bar{\partial}\left(e^{-\frac{i}{2}\phi}\partial e^{\frac{i}{2}\phi} - e^{\frac{i}{2}\phi}\partial e^{-\frac{i}{2}\phi}\right) &= 4i\alpha^2 \sin \phi \quad .\end{aligned}\tag{3.32}$$

As shown in [17] these equations (or a linear combination of them) can be obtained as a dimensional reduction of the equations of motion for noncommutative U(2) SDYM in 2+2 dimensions.

The equations (3.32) can also be derived from an action which consists of the sum of two WZW actions augmented by a cosine potential,

$$S[f, \bar{f}] = S[f] + S[\bar{f}] \quad \text{with} \quad S[f] \equiv S_W[f] - \alpha^2 \int dt dy (f^2 + f^{-2} - 2) \quad , \tag{3.33}$$

with $S_W[f]$ given in (3.20) for $f \equiv e^{\frac{i}{2}\phi}$ in complexified U(1). However, this action cannot be obtained from the SDYM action in 2+2 dimensions by performing the same field parametrization which led to (3.32).

Comparing the actions (3.19) and (3.33) and considering f and \bar{f} as independent U(1) group valued fields we are tempted to formally identify $f \equiv g_+$ and $\bar{f} \equiv g_-$. Doing this, we immediately realize that the two models differ in their interaction term which generalizes the cosine potential. While in (3.33) the fields f and \bar{f} show only self-interaction, the fields g_+ and g_- in (3.19) interact with each other. As we will see in section 5 this makes a big difference when evaluating the S-matrix elements.

We close this section by observing that the equations of motion (3.12) can also be obtained directly in two dimensions by using the bicomplex approach described in [22]. In fact, if instead of (3.31) we choose

$$G = \begin{pmatrix} e^{\frac{i}{2}\phi_+} + e^{-\frac{i}{2}\phi_-} & -i e^{\frac{i}{2}\phi_+} + i e^{-\frac{i}{2}\phi_-} \\ i e^{\frac{i}{2}\phi_+} - i e^{-\frac{i}{2}\phi_-} & e^{\frac{i}{2}\phi_+} + e^{-\frac{i}{2}\phi_-} \end{pmatrix} \tag{3.34}$$

it is easy to prove that (3.29) yields exactly the set of equations (3.12). Therefore, by exploiting the results in [22] it should be straightforward to construct the first nontrivial conserved currents for the present model.

4 Noncommutative solitons

Dressing approach in 2+1 dimensions. The existence of the linear system allows for powerful methods to systematically construct explicit solutions for Ψ and hence for $\Phi^\dagger = \Psi|_{\zeta=0}$ or Υ . For our purposes the so-called dressing method [26, 27] proves to be most practical, and so we shall first present it here for our linear system (2.4), before reducing the results to solitonic solutions of the noncommutative sine-Gordon equations.

The central idea is to demand analyticity in the spectral parameter ζ for the linear system (2.4), which strongly restricts the possible form of Ψ . The most elegant way to exploit this constraint starts from the observation that the left hand sides of the differential relations (D):=(2.5) as well as the reality condition (R):=(2.6) do not depend on ζ while their right hand sides are expected to be nontrivial functions of ζ (except for the trivial case $\Psi = \Psi^0$). More specifically, \mathbb{CP}^1 being compact, the matrix function $\Psi(\zeta)$ cannot be holomorphic everywhere but must possess some poles, and hence the right hand sides of (D) and (R) should display these (and complex conjugate) poles as well. The resolution of this conundrum demands that the residues of the right hand sides at any would-be pole in ζ have to vanish. We are now going to evaluate these conditions.

The dressing method builds a solution $\Psi_N(t, x, y, \zeta)$ featuring N simple poles at positions $\mu_1, \mu_2, \dots, \mu_N$ by left-multiplying an $(N-1)$ -pole solution $\Psi_{N-1}(t, x, y, \zeta)$ with a single-pole factor of the form $(1 + \frac{\mu_N - \bar{\mu}_N}{\zeta - \mu_N} P_N(t, x, y))$, where the $n \times n$ matrix function P_N is yet to be determined. In addition, we are free to right-multiply $\Psi_{N-1}(t, x, y, \zeta)$ with some constant unitary matrix $\widehat{\Psi}_N^0$. Starting from $\Psi_0 = \mathbf{1}$, the iteration $\Psi_0 \mapsto \Psi_1 \mapsto \dots \mapsto \Psi_N$ yields a multiplicative ansatz for Ψ_N which, via partial fraction decomposition, may be rewritten in an additive form (as a sum of simple pole terms). Let us trace this iterative procedure constructively.

In accord with the outline above, the one-pole ansatz must read ($\widehat{\Psi}_1^0 =: \Psi_1^0$)

$$\Psi_1 = \left(\mathbf{1} + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1} P_1 \right) \Psi_1^0 = \left(\mathbf{1} + \frac{\Lambda_{11} S_1^\dagger}{\zeta - \mu_1} \right) \Psi_1^0 \quad (4.1)$$

with some $n \times r_1$ matrix functions Λ_{11} and S_1 for some $1 \leq r_1 < n$. The normalization matrix Ψ_1^0 is constant and unitary. It is quickly checked that

$$\text{res}_{\zeta=\bar{\mu}_1}(R) = 0 \quad \implies \quad P_1^\dagger = P_1 = P_1^2 \quad \implies \quad P_1 = T_1 (T_1^\dagger T_1)^{-1} T_1^\dagger, \quad (4.2)$$

meaning that P_1 is a rank r_1 projector built from an $n \times r_1$ matrix function T_1 . The columns of T_1 span the image of P_1 and obey $P_1 T_1 = T_1$. When using the second parametrization of Ψ_1 in (4.1) one finds that

$$\text{res}_{\zeta=\bar{\mu}_1}(R) = 0 \quad \implies \quad (\mathbf{1} - P_1) S_1 \Lambda_{11}^\dagger = 0 \quad \implies \quad T_1 = S_1 \quad (4.3)$$

modulo a freedom of normalization. Finally, the differential relations yield

$$\text{res}_{\zeta=\bar{\mu}_1}(D) = 0 \quad \implies \quad (\mathbf{1} - P_1) \bar{L}_1^{A,B} (S_1 \Lambda_{11}^\dagger) = 0 \quad \implies \quad \bar{L}_1^{A,B} S_1 = S_1 \Gamma_1^{A,B} \quad (4.4)$$

for some $r_1 \times r_1$ matrices Γ_1^A and Γ_1^B , after having defined

$$\bar{L}_i^A := \partial_u - \bar{\mu}_i \partial_x \quad \text{and} \quad \bar{L}_i^B := \mu_i (\partial_x - \bar{\mu}_i \partial_v) \quad \text{for } i = 1, 2, \dots, N. \quad (4.5)$$

Because the $\bar{L}_i^{A,B}$ are linear differential operators it is easy to write down the general solution for (4.4): Introduce “co-moving coordinates”

$$w_i := x + \bar{\mu}_i u + \bar{\mu}_i^{-1} v \quad \implies \quad \bar{w}_i = x + \mu_i u + \mu_i^{-1} v \quad \text{for } i = 1, 2, \dots, N \quad (4.6)$$

so that on functions of (w_i, \bar{w}_i) alone the $\bar{L}_i^{A,B}$ act as

$$\bar{L}_i^A = \bar{L}_i^B = (\mu_i - \bar{\mu}_i) \frac{\partial}{\partial \bar{w}_i}. \quad (4.7)$$

Hence, (4.4) is solved by

$$S_1(t, x, y) = \widehat{S}_1(w_1) e^{\bar{w}_1 \Gamma_1 / (\mu_1 - \bar{\mu}_1)} \quad \text{for any } w_1\text{-holomorphic } n \times r_1 \text{ matrix function } \widehat{S}_1 \quad (4.8)$$

and $\Gamma_1^A = \Gamma_1^B =: \Gamma_1$. Appearing to the right of \widehat{S}_1 , the exponential factor is seen to drop out in the formation of P_1 via (4.2) and (4.3). Thus, no generality is lost by taking $\Gamma_1 = 0$. We learn that any w_1 -holomorphic $n \times r_1$ matrix T_1 is admissible to build a projector P_1 which then yields a solution Ψ_1 (and thus Φ) via (4.1). Note that Λ_{11} need not be determined separately but follows from our above result. It is not necessary to also consider the residues at $\zeta = \mu_1$ since their vanishing leads merely to the hermitian conjugated conditions.

Let us proceed to the two-pole situation. The dressing ansatz takes the form ($\Psi_1^0 \widehat{\Psi}_2^0 =: \Psi_2^0$)

$$\Psi_2 = \left(\mathbf{1} + \frac{\mu_2 - \bar{\mu}_2}{\zeta - \mu_2} P_2 \right) \left(\mathbf{1} + \frac{\mu_1 - \bar{\mu}_1}{\zeta - \mu_1} P_1 \right) \Psi_2^0 = \left(\mathbf{1} + \frac{\Lambda_{21} S_1^\dagger}{\zeta - \mu_1} + \frac{\Lambda_{22} S_2^\dagger}{\zeta - \mu_2} \right) \Psi_2^0, \quad (4.9)$$

where P_2 and S_2 are to be determined but P_1 and S_1 can be copied from above. Indeed, inspecting the residues of (R) and (D) at $\zeta = \bar{\mu}_1$ simply confirms that

$$P_1 = T_1 (T_1^\dagger T_1)^{-1} T_1^\dagger \quad \text{and} \quad T_1 = S_1 \quad \text{with} \quad S_1 = \widehat{S}_1(w_1) \quad (4.10)$$

is just carried over from the one-pole solution. Relations for P_2 and S_2 arise from

$$\text{res}_{\zeta=\bar{\mu}_2}(R) = 0 \implies (\mathbf{1} - P_2) P_2 = 0 \implies P_2 = T_2 (T_2^\dagger T_2)^{-1} T_2^\dagger, \quad (4.11)$$

$$\text{res}_{\zeta=\bar{\mu}_2}(R) = 0 \implies \Psi_2(\bar{\mu}_2) S_2 \Lambda_{22}^\dagger = (\mathbf{1} - P_2) \left(1 - \frac{\mu_1 - \bar{\mu}_1}{\mu_1 - \mu_2} P_1 \right) S_2 \Lambda_{22}^\dagger = 0, \quad (4.12)$$

where the first equation makes use of the multiplicative form of the ansatz (4.9) while the second one exploits the additive version. We conclude that P_2 is again a hermitian projector (of some rank r_2) and thus built from an $n \times r_2$ matrix function T_2 . Furthermore, (4.12) reveals that T_2 cannot be identified with S_2 this time, but we rather have

$$T_2 = \left(1 - \frac{\mu_1 - \bar{\mu}_1}{\mu_1 - \mu_2} P_1 \right) S_2 \quad (4.13)$$

instead. Finally, we consider

$$\text{res}_{\zeta=\bar{\mu}_2}(D) = 0 \implies \Psi_2(\bar{\mu}_2) \bar{L}_2^{A,B} (S_2 \Lambda_{22}^\dagger) = 0 \implies \bar{L}_2^{A,B} S_2 = S_2 \Gamma_2^{A,B} \quad (4.14)$$

which is solved by

$$S_2(t, x, y) = \widehat{S}_2(w_2) e^{\bar{w}_2 \Gamma_2 / (\mu_2 - \bar{\mu}_2)} \quad \text{for any } w_2\text{-holomorphic } n \times r_2 \text{ matrix function } \widehat{S}_2 \quad (4.15)$$

and $\Gamma_2^A = \Gamma_2^B =: \Gamma_2$. Once more, we are entitled to put $\Gamma_2 = 0$. Hence, the second pole factor in (4.9) is constructed in the same way as the first one, except for the small complication (4.13). Again, Λ_{21} and Λ_{22} can be read off the result if needed.

It is now clear how the iteration continues. After N steps the final result reads

$$\Psi_N = \left\{ \prod_{\ell=0}^{N-1} \left(\mathbf{1} + \frac{\mu_{N-\ell} - \bar{\mu}_{N-\ell}}{\zeta - \mu_{N-\ell}} P_{N-\ell} \right) \right\} \Psi_N^0 = \left\{ \mathbf{1} + \sum_{i=1}^N \frac{\Lambda_{Ni} S_i^\dagger}{\zeta - \mu_i} \right\} \Psi_N^0, \quad (4.16)$$

featuring hermitian rank r_i projectors P_i at $i = 1, 2, \dots, N$, via

$$P_i = T_i (T_i^\dagger T_i)^{-1} T_i^\dagger \quad \text{with} \quad T_i = \left\{ \prod_{\ell=1}^{i-1} \left(\mathbf{1} - \frac{\mu_{i-\ell} - \bar{\mu}_{i-\ell}}{\mu_{i-\ell} - \bar{\mu}_i} P_{i-\ell} \right) \right\} S_i, \quad (4.17)$$

where

$$S_i(t, x, y) = \widehat{S}_i(w_i) \quad (4.18)$$

for arbitrary w_i -holomorphic $n \times r_i$ matrix functions $\widehat{S}_i(w_i)$. The corresponding classical Yang and Leznov fields are

$$\Phi_N = \Psi_N^\dagger(\zeta=0) = \Psi_N^{0\dagger} \prod_{i=1}^N (\mathbf{1} - \rho_i P_i) \quad \text{with} \quad \rho_i = 1 - \frac{\mu_i}{\bar{\mu}_i}, \quad (4.19)$$

$$\Upsilon_N = \lim_{\zeta \rightarrow \infty} \zeta (\Psi_N(\zeta) \Psi_N^{0\dagger} - \mathbf{1}) = \sum_{i=1}^N (\mu_i - \bar{\mu}_i) P_i. \quad (4.20)$$

The solution space constructed here is parametrized (slightly redundantly) by the set $\{\widehat{S}_i\}_1^N$ of matrix-valued holomorphic functions and the pole positions μ_i . The so-constructed classical configurations have solitonic character (meaning finite energy) when all these functions are algebraic.

The dressing technique as presented above is well known in the commutative theory; novel is only the realization that it carries over verbatim to the noncommutative situation by simply understanding all products as star products (and likewise inverses, exponentials, etc.). Of course, it may be technically difficult to \star -invert some matrix, but one may always fall back on an expansion in powers of θ .

Solitons of the noncommutative sine-Gordon theory. We should now be able to generate N -soliton solutions to the noncommutative sine-Gordon equations, say (3.17), by applying the reduction from 2+1 to 1+1 dimensions (see previous section) to the above strategy for the group $U(2)$, i.e. putting $n=2$. In order to find nontrivial solutions, we specify the constant matrix \mathcal{E} in the ansatz (3.1) for Ψ as

$$\mathcal{E} = e^{-i\frac{\pi}{4}\sigma_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad (4.21)$$

which obeys the relations $\mathcal{E}\sigma_3 = \sigma_1\mathcal{E}$ and $\mathcal{E}\sigma_1 = -\sigma_3\mathcal{E}$. Pushing \mathcal{E} beyond V we can write

$$\Phi(t, x, y) = W(x) \widetilde{g}(u, v) W^\dagger(x) \quad \text{with} \quad W(x) = e^{-i\alpha x \sigma_3} \quad (4.22)$$

and

$$\widetilde{g}(u, v) = \mathcal{E} g(u, v) \mathcal{E}^\dagger = \mathcal{E} \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} \mathcal{E}^\dagger = \frac{1}{2} \begin{pmatrix} g_+ + g_- & g_+ - g_- \\ g_+ - g_- & g_+ + g_- \end{pmatrix} . \quad (4.23)$$

With hindsight from the commutative case [27] we choose

$$\widehat{\Psi}_i^0 = \sigma_3 \quad \forall i \quad \Longleftrightarrow \quad \Psi_N^0 = \sigma_3^N \quad (4.24)$$

(which commutes with W) and restrict the poles of Ψ to the imaginary axis, $\mu_i = i p_i$ with $p_i \in \mathbb{R}$. Therewith, the co-moving coordinates (4.6) become

$$w_i = x - i(p_i u - p_i^{-1} v) =: x - i\eta_i(u, v) \quad , \quad (4.25)$$

defining η_i as real linear functions of the light-cone coordinates. Consequentially, from (4.19) we get $\rho_i = 2$ and find that

$$\widetilde{g}_N(u, v) = \sigma_3^N \prod_{i=1}^N (1 - 2\widetilde{P}_i(u, v)) \quad \text{with} \quad P_i = W \widetilde{P}_i W^\dagger . \quad (4.26)$$

Repeating the analysis of the previous subsection, one is again led to construct hermitian projectors

$$\widetilde{P}_i = \widetilde{T}_i (\widetilde{T}_i^\dagger \widetilde{T}_i)^{-1} \widetilde{T}_i^\dagger \quad \text{with} \quad \widetilde{T}_i = \prod_{\ell=1}^{i-1} \left(1 - \frac{2p_{i-\ell}}{p_{i-\ell} + p_i} \widetilde{P}_{i-\ell} \right) \widetilde{S}_i \quad , \quad (4.27)$$

where 2×1 matrix functions $\widetilde{S}_i(u, v)$ are subject to

$$\widetilde{L}_i^{A,B} \widetilde{S}_i = \widetilde{S}_i \widetilde{\Gamma}_i \quad \text{for} \quad i = 1, 2, \dots, N \quad (4.28)$$

and some numbers $\tilde{\Gamma}_i$ (note that now rank $r_i=1$) which again we can put to zero. On functions of the reduced co-moving coordinates η_i alone,

$$\tilde{L}_i^{A,B} = W^\dagger \bar{L}_i^{A,B} W = (\mu_i - \bar{\mu}_i) W^\dagger \frac{\partial}{\partial \bar{w}_i} W = p_i \left(\frac{\partial}{\partial \eta_i} + \alpha \sigma_3 \right) \quad (4.29)$$

so that (4.28) is solved by

$$\tilde{S}_i(u, v) = \hat{\tilde{S}}_i(\eta_i) = \begin{pmatrix} \gamma_{i1} e^{-\alpha \eta_i} \\ i \gamma_{i2} e^{+\alpha \eta_i} \end{pmatrix} = e^{-\alpha \eta_i \sigma_3} \begin{pmatrix} \gamma_{i1} \\ i \gamma_{i2} \end{pmatrix} \quad \text{with } \gamma_{i1}, \gamma_{i2} \in \mathbb{C} \quad . \quad (4.30)$$

Furthermore, it is useful to rewrite

$$\gamma_{i1} \gamma_{i2} =: \lambda_i^2 \quad \text{and} \quad \gamma_{i2}/\gamma_{i1} =: \gamma_i^2 \quad \Longleftrightarrow \quad \begin{pmatrix} \gamma_{i1} \\ i \gamma_{i2} \end{pmatrix} = \lambda_i \begin{pmatrix} \gamma_i^{-1} \\ i \gamma_i \end{pmatrix} \quad (4.31)$$

because then $|\gamma_i|$ may be absorbed into η_i by shifting $\alpha \eta_i \mapsto \alpha \eta_i + \ln |\gamma_i|$. The multipliers λ_i drop out in the computation of \tilde{P}_i . Finally, to make contact with the form (4.23) we restrict the constants γ_i to be real.

Let us check the one-soliton solution, i.e. put $N=1$. Suppressing the indices momentarily, absorbing γ into η and dropping λ , we infer that

$$\tilde{T} = \begin{pmatrix} e^{-\alpha \eta} \\ i e^{\alpha \eta} \end{pmatrix} \implies \tilde{P} = \frac{1}{2 \operatorname{ch} 2\alpha \eta} \begin{pmatrix} e^{-2\alpha \eta} & -i \\ i & e^{+2\alpha \eta} \end{pmatrix} \implies \tilde{g} = \begin{pmatrix} \operatorname{th} 2\alpha \eta & \frac{i}{\operatorname{ch} 2\alpha \eta} \\ \frac{i}{\operatorname{ch} 2\alpha \eta} & \operatorname{th} 2\alpha \eta \end{pmatrix} \quad (4.32)$$

which has $\det \tilde{g} = 1$. Since here the entire coordinate dependence comes in the single combination $\eta(u, v)$, all star products trivialize and the one-soliton configuration coincides with the commutative one. Hence, the field ρ drops out, $\tilde{g} \in \operatorname{SU}(2)$, and we find, comparing (4.32) with (4.23), that

$$\frac{1}{2}(g_+ + g_-) = \cos \frac{\varphi}{2} = \operatorname{th} 2\alpha \eta \quad \text{and} \quad \frac{1}{2i}(g_+ - g_-) = \sin \frac{\varphi}{2} = \frac{1}{\operatorname{ch} 2\alpha \eta} \quad (4.33)$$

which implies

$$\tan \frac{\varphi}{4} = e^{-2\alpha \eta} \implies \varphi = 4 \arctan e^{-2\alpha \eta} = -2 \arcsin(\operatorname{th} 2\alpha \eta) \quad , \quad (4.34)$$

reproducing the well known sine-Gordon soliton with mass $m = 2\alpha$. Its moduli parameters are the velocity $\nu = \frac{1-p^2}{1+p^2}$ and the center of inertia $y_0 = \frac{1}{\alpha} \sqrt{1-\nu^2} \ln |\gamma|$ at zero time [27]. In passing we note that in the “ h description” the soliton solution takes the form

$$h_1 = p \operatorname{th} 2\alpha \eta \quad \text{and} \quad h_2 = \frac{p}{\operatorname{ch} 2\alpha \eta} \implies h = p \operatorname{th}(\alpha \eta + \frac{i\pi}{4}) = p e^{\frac{i}{2}\varphi} \quad . \quad (4.35)$$

Noncommutativity becomes relevant for multi-solitons. At $N=2$, for instance, one has

$$\tilde{g}_2 = (1 - 2\tilde{P}_1)(1 - 2\tilde{P}_2) \quad \text{with} \quad \tilde{P}_1 = \tilde{P} \quad \text{from (4.32)} \quad \text{and} \quad \tilde{P}_2 = \tilde{T}_2 (\tilde{T}_2^\dagger \tilde{T}_2)^{-1} \tilde{T}_2^\dagger \quad (4.36)$$

$$\text{where} \quad \tilde{T}_2 = \left(\mathbf{1} - \frac{2p_1}{p_1 + p_2} \tilde{P}_1 \right) \hat{\tilde{S}}_2 \quad \text{and} \quad \hat{\tilde{S}}_2 = e^{-\alpha \eta_2 \sigma_3} \begin{pmatrix} \gamma_2^{-1} \\ i \gamma_2 \end{pmatrix} \quad \text{with} \quad \gamma_2 \in \mathbb{R} \quad .$$

We refrain from writing down the lengthy explicit expression for \tilde{g}_2 in terms of the noncommuting coordinates η_1 and η_2 , but one cannot expect to find a unit (star-)determinant for \tilde{g}_2 except in the commutative limit. This underscores the necessity of extending the matrices to $\operatorname{U}(2)$ and the inclusion of a nontrivial ρ at the multi-soliton level.

It is not surprising that the just-constructed noncommutative sine-Gordon solitons themselves descend directly from BPS solutions of the 2+1 dimensional integrable sigma model. Indeed, putting back the x dependence via (4.22), the 2+1 dimensional projectors P_i are built from 2×1 matrices

$$S_i = W(x) \widehat{S}_i(\eta_i) = e^{-i\alpha w_i \sigma_3} \begin{pmatrix} \gamma_i^{-1} \\ i\gamma_i \end{pmatrix} = \begin{pmatrix} 1 \\ i\gamma_i^2 e^{2i\alpha w_i} \end{pmatrix} \gamma_i^{-1} e^{-i\alpha w_i} . \quad (4.37)$$

In the last expression the right factor drops out on the computation of projectors; the remaining column vector agrees with the standard conventions [23, 8, 27, 9]. Reassuringly, the coordinate dependence has combined into w_i . The ensuing 2+1 dimensional configurations Φ_N are nothing but noncommutative multi-plane-waves the simplest examples of which were already investigated in [9].

5 Tree amplitudes

In this section we compute tree-level amplitudes for the noncommutative generalization of the sine-Gordon model proposed in section 3, both in the Yang and the Leznov formulation. In commutative geometry the sine-Gordon S-matrix factorizes in two-particle processes and no particle production occurs, as a consequence of the existence of an infinite number of conservation laws. In the noncommutative case it is interesting to investigate whether the presence of an infinite number of conserved currents is still sufficient to guarantee the integrability of the system in the sense of having a factorized S-matrix.

A previous noncommutative version of the sine-Gordon model with an infinite set of conserved currents was proposed in [22], and its S-matrix was studied in [17]. Despite the existence of an infinite chain of conservation laws, it turned out that particle production occurs in this model and that the S-matrix is neither factorized nor causal.⁹ As already stressed in section 3, the noncommutative generalization of the sine-Gordon model we propose in this paper differs from the one studied in [22] in the generalization of the cosine potential. Therefore, both theories describe the dynamics of two real scalar fields, but the structure of the interaction terms between the two fields is different. We then expect the scattering amplitudes of the present theory to behave differently from those of the previous one. To this end we will compute the amplitudes corresponding to $2 \rightarrow 2$ processes for the fields ρ and φ in the g -model (Yang formulation) as well as for the fields h_1 and h_2 in the h -model (Leznov formulation). In the g -model we will also compute $2 \rightarrow 4$ and $3 \rightarrow 3$ amplitudes for the massive field φ . In both models the S-matrix will turn out to be *factorized* and *causal* in spite of their time-space noncommutativity.

Amplitudes in the “ g -model”. Feynman rules. We parametrize the g -model with (ρ, φ) as in (3.21) since in this parametrization the mass matrix turns out to be diagonal, with zero mass for ρ and $m=2\alpha$ for φ . Expanding the action (3.21) up to the fourth order in the fields, we read off the following Feynman rules:

- The propagators

$$\text{—————} \equiv \langle \varphi \varphi \rangle = \frac{2i}{k^2 - 4\alpha^2} , \quad (5.1)$$

$$\text{.....} \equiv \langle \rho \rho \rangle = \frac{2i}{k^2} . \quad (5.2)$$

⁹Acausal behaviour in noncommutative field theory was first observed in [18] and shown to be related to time-space noncommutativity.

$$\begin{array}{cc}
\begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} = 2i\alpha^2 \cos^2(\theta Ep) \quad , & \begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} = 0 \quad , \\
\begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} = -\frac{i}{2}p^2 \sin^2(\theta Ep) \quad , & \begin{array}{c} 1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} 2 \\ \diagup \\ 4 \end{array} = \frac{i}{2}E^2 \sin^2(\theta Ep) \quad .
\end{array}$$

The second diagram is actually affected by a collinear divergence since the total momentum $k_1 + k_4$ for the internal massless particle is on-shell vanishing. We regularize this divergence by temporarily giving a small mass to the ρ particle. It is easy to see that the amplitude is zero for any value of the small mass since the wedge products $k_1 \wedge k_4$ and $k_2 \wedge k_3$ from the two vertices always vanish. As an alternative procedure we can put one of the external particles slightly off-shell, so obtaining a finite result which vanishes in the on-shell limit.

Summing all the contributions, for the $\varphi\varphi \rightarrow \varphi\varphi$ amplitude we arrive at

$$A_{\varphi\varphi \rightarrow \varphi\varphi} = 2i\alpha^2 \quad , \quad (5.11)$$

which perfectly describes a *causal* amplitude.

A nonvanishing $\varphi\varphi \rightarrow \varphi\varphi$ amplitude appears also in the noncommutative sine-Gordon proposal of [22, 17]. However, there the amplitude has a nontrivial θ -dependence which is responsible for acausal behavior. Comparing the present result with the result in [17], we observe that the same kind of diagrams contribute. The main difference is that the exchanged particle is now massless instead of massive. This crucial difference leads to the cancellation of the θ -dependent trigonometric behaviour which in the previous case gave rise to acausality.

Amplitude $\rho\rho \rightarrow \rho\rho$. In this case the center-of-mass momenta are given by

$$k_1 = (E, E) \quad , \quad k_2 = (E, -E) \quad , \quad k_3 = (-E, E) \quad , \quad k_4 = (-E, -E) \quad , \quad (5.12)$$

where the on-shell condition $E^2 - p^2 = 0$ has already been taken into account. For this amplitude we have the following contributions

$$\begin{array}{cc}
\begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} = 0 \quad , & \begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} = 0 \quad , \\
\begin{array}{c} 1 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 3 \end{array} = -\frac{i}{2}E^2 \sin^2(\theta E^2) \quad , & \begin{array}{c} 1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} 2 \\ \diagup \\ 4 \end{array} = \frac{i}{2}E^2 \sin^2(\theta E^2) \quad .
\end{array}$$

Again, a collinear divergence appears in the second diagram. In order to regularize the divergence we can proceed as before by assigning a small mass to the ρ particle. The main difference with

respect to the previous case is that now the ρ particle also appears as an external particle, with the consequence that the on-shell momenta in (5.12) will get modified by the introduction of a regulator mass. A careful calculation shows that the amplitude is zero for any value of the regulator mass, due to the vanishing of the factors $k_1 \wedge k_4$ and $k_2 \wedge k_3$ from the vertices.

Therefore, the two nonvanishing contributions add to

$$A_{\rho\rho \rightarrow \rho\rho} = 0 \quad . \quad (5.13)$$

Amplitude $\varphi\rho \rightarrow \varphi\rho$. There are two possible configurations of momenta in the center-of-mass frame, describing the scattering of the massive particle with either a left-moving or a right-moving massless one. In the left-moving case the momenta are

$$k_1 = (E, p) \ , \quad k_2 = (p, -p) \ , \quad k_3 = (-E, p) \ , \quad k_4 = (-p, -p) \ , \quad (5.14)$$

while in the right-moving case we have

$$k_1 = (E, -p) \ , \quad k_2 = (p, p) \ , \quad k_3 = (-E, p) \ , \quad k_4 = (-p, -p) \ . \quad (5.15)$$

For the left-moving case (5.14) the results are

$$\begin{aligned} \text{Diagram 1} &= -\frac{i}{2}Ep \sin(\theta Ep) \sin(\theta p^2) \ , \\ \text{Diagram 2} &= \frac{i}{2}Ep \sin(\theta Ep) \sin(\theta p^2) \ , \\ \text{Diagram 3} &= 0 \ , \quad \text{Diagram 4} = 0 \ . \end{aligned}$$

For the right-moving choice (5.15), we obtain instead

$$\begin{aligned} \text{Diagram 1} &= 0 \ , \quad \text{Diagram 2} = 0 \ , \\ \text{Diagram 3} &= 0 \ , \quad \text{Diagram 4} = 0 \ . \end{aligned}$$

In this second case an infrared divergence is present due to the massless propagator, but again it can be cured as described before. In both cases the scattering amplitude vanishes,

$$A_{\varphi\rho \rightarrow \varphi\rho} = 0 \quad . \quad (5.16)$$

Amplitude $\varphi\varphi \rightarrow \rho\rho$. The momenta in the center-of-mass frame are given by

$$k_1 = (E, p) , \quad k_2 = (E, -p) , \quad k_3 = (-E, E) , \quad k_4 = (-E, -E) . \quad (5.17)$$

In this case we have three kinds of diagrams contributing. The corresponding results are

$$\begin{aligned} \begin{array}{c} \text{Diagram 1: } \begin{array}{c} \text{Solid lines } 1, 2 \text{ meet at a vertex.} \\ \text{Dashed lines } 3, 4 \text{ meet at another vertex.} \\ \text{A dashed line connects the two vertices.} \end{array} &= \frac{i}{2} E p \sin(\theta E p) \sin(\theta E^2) , \\ \begin{array}{c} \text{Diagram 2: } \begin{array}{c} \text{Solid lines } 1, 2 \text{ meet at a vertex.} \\ \text{Dashed lines } 3, 4 \text{ meet at another vertex.} \\ \text{A dashed line connects the two vertices.} \end{array} &= -\frac{i}{2} E p \sin(\theta E p) \sin(\theta E^2) , \\ \begin{array}{c} \text{Diagram 3: } \begin{array}{c} \text{Solid lines } 1, 2 \text{ meet at a vertex.} \\ \text{Dashed lines } 3, 4 \text{ meet at another vertex.} \\ \text{A solid line connects the two vertices.} \end{array} &= 0 , \quad \begin{array}{c} \text{Diagram 4: } \begin{array}{c} \text{Solid lines } 1, 2 \text{ meet at a vertex.} \\ \text{Dashed lines } 3, 4 \text{ meet at another vertex.} \\ \text{A solid line connects the two vertices.} \end{array} &= 0 . \end{aligned}$$

Summing the four contributions, we obtain

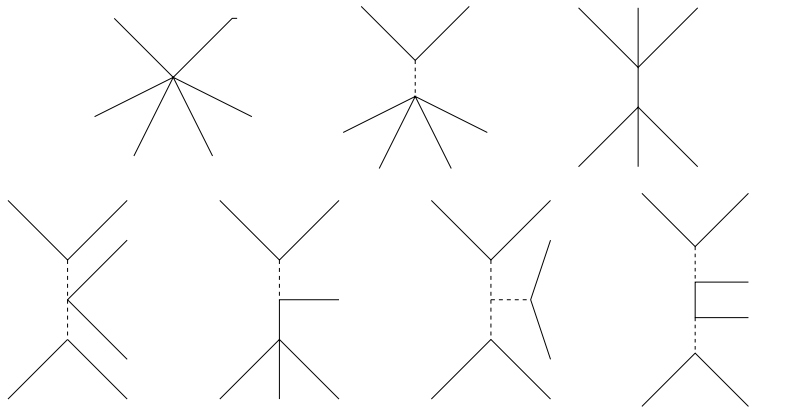
$$A_{\varphi\varphi \rightarrow \rho\rho} = 0 \quad (5.18)$$

as it should be expected for a production amplitude in an integrable model. The same is true for the time-reversed production,

$$A_{\rho\rho \rightarrow \varphi\varphi} = 0 . \quad (5.19)$$

Summarizing, we have found that the only nonzero amplitude for tree-level $2 \rightarrow 2$ processes is the one describing the scattering among two of the massive excitations. The result is constant, independent of the momenta and so describes a perfectly *causal* process. Since the result is independent of the noncommutation parameter θ it agrees with the four-point amplitude for the ordinary sine-Gordon model. Finally, we have found that the production amplitudes $\varphi\varphi \rightarrow \rho\rho$ and $\rho\rho \rightarrow \varphi\varphi$ vanish, as required for ordinary integrable theories.

As a further check of our calculation and an additional test of our model we have computed the production amplitude $\varphi\varphi \rightarrow \varphi\varphi\varphi\varphi$ and the scattering amplitude $\varphi\varphi\varphi \rightarrow \varphi\varphi\varphi$. In both cases the topologies we have to consider are



Due to the growing number of channels and ordering of vertices, it is no longer practical to perform the calculations by hand. We have used *Mathematica*® to symmetrize the vertices and take automatically into account the different diagrams obtained by exchanging momenta entering a given vertex. The computation has been performed with assigned values of the external momenta but arbitrary values for α^2 and θ . We have found a vanishing result for both the scattering and the production amplitude. This is in agreement with the commutative sine-Gordon model results.

Amplitudes in the “ h -model”. We now discuss the $2 \rightarrow 2$ amplitudes in the Leznov formulation. The theory is again described by two interacting fields, h_1 massless and h_2 massive. Referring to the action (3.28) we extract the following Feynman rules,

- The propagators

$$\text{-----} \equiv \langle h_1 h_1 \rangle = \frac{i}{2k^2} \quad , \quad (5.20)$$

$$\text{—————} \equiv \langle h_2 h_2 \rangle = \frac{i/2}{k^2 - 4\alpha^2} \quad . \quad (5.21)$$

- The vertex

$$\begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 3 \end{array} = -4\alpha (k_{3t} - k_{3y}) F(k_1, k_2, k_3) \quad . \quad (5.22)$$

Again, we compute scattering amplitudes in the center-of-mass frame. Given the particular structure of the vertex, at tree level there is no $h_1 h_1 \rightarrow h_1 h_1$ scattering. To find the $h_2 h_2 \rightarrow h_2 h_2$ amplitude we assign the momenta (5.10) to the external particles. The contributions are

$$\begin{array}{ccc} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} & = & -16i\alpha^2 \cos^2(\theta Ep) \quad , \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \quad 3 \end{array} & = & 16i\alpha^2 \cos^2(\theta Ep) \quad , \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \quad 3 \end{array} & = & 0 \quad . \end{array}$$

We note that a collinear divergence appears in the last diagram which can be regularized as described before. Summing the two nonvanishing contributions we obtain complete cancellation.

For the $h_2 h_2 \rightarrow h_1 h_1$ amplitude the center-of-mass-momenta are given in (5.17). The only topology contributing to this production amplitude has two channels, yielding

$$\begin{array}{ccc} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \quad 3 \end{array} & = & 0 \quad , \\ \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} & = & 0 \quad , \end{array}$$

which are both zero, so giving a vanishing result once more. The same is true for the $h_1 h_1 \rightarrow h_2 h_2$ production process.

Finally, for the $h_1 h_2 \rightarrow h_1 h_2$ amplitude, we refer to the center-of-mass momenta defined in (5.14) and (5.15). In both cases the contributions are

$$\text{Diagram 1} = 0, \quad \text{Diagram 2} = 0,$$

and so we find that the sum of the two channels is always equal to zero.

Since all the $2 \rightarrow 2$ amplitudes vanish, the S-matrix is trivially causal and factorized.

Both in the ordinary and noncommutative cases the “ h -model” is dual to the “ g -model”. In the commutative limit the “ g -model” gives rise to a sine-Gordon model plus a free field which can be set to zero. In this limit our amplitudes exactly reproduce the sine-Gordon amplitudes. On the other hand, the amplitudes for the “ h -model” all vanish. Therefore, in the commutative limit they do not reproduce anything immediately recognizable as an ordinary sine-Gordon amplitude. This can be understood by observing that, both in the ordinary and in the noncommutative case, the Leznov formulation is an alternative description of the sine-Gordon dynamics and obtained from the standard Yang formulation by the *nonlocal field redefinition* given in (3.27). Therefore, it is expected that the scattering amplitudes for the elementary excitations, which are different in the two formulations, do not resemble each other.

6 Conclusions

We have proposed a novel noncommutative sine-Gordon system based on *two* scalar fields, which seems to retain all advantages of 1+1 dimensional integrable models known from the commutative limit. The rationale for introducing a second scalar field was provided by deriving the sine-Gordon equations and action through dimensional and algebraic reduction of an integrable 2+1 dimensional sigma model: In the noncommutative extension of this scheme it is natural to generalize the algebraic reduction of $SU(2) \rightarrow U(1)$ to one of $U(2) \rightarrow U(1) \times U(1)$. We gave two Yang-type and one Leznov-type parametrizations of the coupled system in (3.12), (3.17) and (3.26) and provided the actions for them, including a comparison with previous proposals. It was then outlined how to explicitly construct noncommutative sine-Gordon multi-solitons via the dressing method based on the underlying linear system. We found that the one-soliton configuration agrees with the commutative one but already the two-soliton solutions gets Moyal deformed.

What is the gain of doubling the field content as compared to the standard sine-Gordon system or its straightforward star deformation? Usually, time-space noncommutativity adversely affects the causality and unitarity of the S-matrix (see, e.g. [21, 22, 17]), even in the presence of an infinite number of local conservation laws. In contrast, the model described here seems to possess an S-matrix which is *causal* and *factorized*, as we checked for all tree-level $2 \rightarrow 2$ processes both in the Yang and Leznov formulations. Furthermore, we verified the vanishing of some $3 \rightarrow 3$ scattering amplitudes and $2 \rightarrow 4$ production amplitudes thus proving the absence of particle production.

It would be nice to understand what actually drives a system to be integrable in the noncommutative case. A hint in this direction might be that the model proposed in [22] has been constructed directly in two dimensions even if its equations of motion (but *not* the action) can be obtained by a suitable reduction of a four dimensional system (noncommutative self-dual Yang-Mills). The

model proposed in this paper, instead, originates directly, already at the level of the action, from the reduction of noncommutative self-dual Yang-Mills theory which is known to be integrable and related to the $N=2$ string [2].

Several directions of future research are suggested by our results. First, one might hope that our noncommutative two-field sine-Gordon model is equivalent to some two-fermion model via noncommutative bosonization. Second, it would be illuminating to derive the exact two-soliton solution and extract its scattering properties, either directly in our model or by reducing wave-like solutions of the 2+1 dimensional sigma model [9, 10]. Third, there is no obstruction against applying the ideas and techniques of this paper to other 1+1 dimensional noncommutative integrable systems in order to cure their pathologies as well.

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